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On Maximal Circuits in Directed Graphs

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An exact bound is obtained for the number of edges in a directed graph which ensures the existence of a circuit exceeding a prescribed length.

Another proof of an analogous result of Erdös and Gallai for undirected graphs is supplied in the Appendix.

In [2] the following result was established.

THEOREM (Erdös and Gallai). *Let c be some integer > 1. If there are more than $\frac{1}{2}(n-1)c$ edges in a graph of order n, then the graph contains a cycle of length greater than c.*

In this paper we wish to show the following analogous result for directed graphs.

THEOREM 1. *Let n and c be positive integers and let r be the least nonnegative residue of n modulo c. Let G be a directed 1-graph of order n and without loops. If there are more than $F(n, c) = \frac{1}{2}[n(n+c-2) - r(c-r)]$ edges in G, then G contains a circuit of length greater than c.*

For every n and c there is a 1-graph of order n having $F(n, c)$ edges and no circuits of length greater than c.

In conclusion we wish to give another proof of the theorem of Erdös and Gallai. Proofs were also given by Bondy [5] and Woodall [6].

$|S|$ will denote the number of elements of the set S. We shall denote by *strong component* a maximal strongly connected subgraph. All graphs here discussed are without loops and except for those discussed in the Appendix they are all directed. The *degree* $d(x)$ of a vertex x is the sum of its indegree and its outdegree. We shall use the following result of Ghouila-Houri [3] (see also [1, p. 196]).

THEOREM GH. *Let G be a strongly connected 1-graph of order n and* *without loops. If for each vertex x we have $d(x) \geqslant n$, then G has a Hamiltonian circuit.*

Proof of Theorem 1. For $n \leqslant c$ the theorem is trivially true. Let $n > c$ and assume the theorem to be true for $n' < n$.

It may be easily calculated that

$$F(n, c) - F(n-1, c) = \begin{cases} n + c - 2 & \text{if } r = 0, \\ n + r - 2 & \text{if } r > 0. \end{cases}$$

In any case we have $F(n, c) - F(n-1, c) \geqslant n - 1$.

Let there be a vertex x_0 with $d(x_0) < n$. Put $G = (X, E)$. Consider $G' = G \backslash X_0 = (X \backslash x_0, E')$. Then

$$|E'| = |E| - d(x_0) > F(n, c) - n + 1 \geqslant F(n-1, c)$$

and hence, by the induction hypothesis for $n - 1$, there is a circuit of length greater than c.

We now assume that $d(x) \geqslant n$ for every vertex x of G.

Let $g = (Y, L)$ be a strong component of G. Put $|Y| = m$, $n - m = m'$. We now have

Case 1. $g = G$. Then G is strongly connected. Besides we have $d(x) \geqslant n$ for every x in X. It then follows from Theorem GH that G has a Hamiltonian circuit. Since $n > c$, we have proved Case 1.

Case 2. $g \neq G$. Then $m \neq n$, so that $0 < m < n$, and hence also $0 < m' < n$. Put $X \backslash Y = Z$. Let $\gamma = (Z, L')$ be the subgraph of G induced by Z. Consider a pair of vertices $\{y, z\}$, $y \in Y$, $z \in Z$. If both (y, z) and (z, y) were in E, then we could add z to g and it would still be strongly connected, contradicting the maximality property of g. Then either (y, z) or (z, y) is not in E. It follows that $|E| \leqslant |L| + |L'| + |Y \times Z|$. Assuming that neither g nor γ has a circuit of length $> c$ and applying the induction hypothesis to both g and γ we obtain $|L| \leqslant F(m, c)$, $|L'| \leqslant F(m', c)$; we also have $|Y \times Z| = mm'$.

Let r_1, r_2 be the least nonnegative residues modulo c of m and m' respectively. Put $r = r_0$, $c - r_i = r_i'$ for $i = 0, 1, 2$. $m + m' = n$ implies $r_1 + r_2 \equiv r_0 \pmod{c}$, so that $r_1 + r_2 = (r_0 \text{ or } c + r_0)$ implying that either

(i) $r_1 + r_2 = r_0$, or

(ii) $r_1' + r_2' = r_0'$.

We show that in any case we have

$$r_0 r_0' \leqslant r_1 r_1' + r_2 r_2'. \tag{1}$$

Let (i) hold. Then $r_0r_0' = r_1r_0' + r_2r_0' \leqslant r_1r_1' + r_2r_2'$. If (ii) holds, then by interchanging the roles of r_i and r_i' we arrive again at (1). We now have

$$\begin{aligned}2 \mid E \mid &\leqslant 2[F(m, c) + F(m', c) + mm'] \\ &= m(m + c - 2) + m'(m' + c - 2) \\ &\quad + 2mm' - r_1(c - r_1) - r_2(c - r_2) \\ &= n(n + c - 2) - r_1r_1' - r_2r_2' \\ &\leqslant n(n + c - 2) - r_0r_0' = 2F(n, c),\end{aligned}$$

a contradiction. This proves Theorem 1.

In order to see that the theorem cannot be improved, consider two positive integers n, c. Put $n = kc + r$, $k \geqslant 0$, $0 \leqslant r < c$. Define a graph $G = (X, E)$ of order n in the following way.

$$X = \{x_{ij}\colon i = i_0\,,\ 1 \leqslant i_0 \leqslant k + 1,\ i_0 \text{ arbitrary but fixed},\ 1 \leqslant j \leqslant r;\ \text{or}\ i \neq i_0\,,\ 1 \leqslant i \leqslant k + 1,\ 1 \leqslant j \leqslant c\},$$

$$E = \{(x_{ij}\,, x_{i_1j_1})\colon i \leqslant i_1\}.$$

It is clear that the graph so defined has no circuits of length greater than c. A rather simple computation shows that $\mid E \mid = F(n, c)$. It also follows easily from the proof that the graph so constructed is essentially unique for given n and c.

Theorem 1 admits of a simple corollary.

COROLLARY 1. *If* $\mid E \mid > (n - 1)^2$, *then* $G = (X, E)$ *is Hamiltonian.*

For 1-graphs which are known to be strongly connected, Corollary 1 may be improved. But first we wish to state another result. Let "*G is pan-Hamiltonian*" mean that there is a Hamiltonian path from any vertex to any other vertex in G. A pan-Hamiltonian graph is clearly Hamiltonian.

We now have

THEOREM 2. *Let* $G = (X, E)$ *be a* 1-*graph of order* $n > 2$. *If* $\mid E \mid \geqslant n^2 - 2n + 3$, *then* G *is pan-Hamiltonian.*

Proof. Let $\mid E \mid = n^2 - 2n + 3 = f(n)$. The theorem is clearly true for $n = 3$. Let the theorem be true for $2 < n' < n$. Since G is not complete, there is a vertex x_0 such that $d(x_0) \leqslant 2n - 3$. By deleting the vertex we get $\mid E(G\backslash x_0)\mid \geqslant n^2 - 4n + 6 = f(n - 1)$, so that by the induction hypothesis $G\backslash x_0$ is pan-Hamiltonian. On the other hand it is quite easy to derive from $\mid E \mid$ that for all $x \in X$ we have $d(x) \geqslant n + 1$. Then $d^+(x) \geqslant 2$ for every vertex of G. Let a, b be two arbitrary vertices of G. We now have

Case 1. $d(a) < 2(n-1)$. Since $d^+(a) \geqslant 2$, there is a vertex $y \neq b$ such that $(a, y) \in E$. But $G\backslash a$ is pan-Hamiltonian, so that there is a Hamiltonian path $h(y,, b)$ in $G\backslash a$. Then there is a Hamiltonian path $h(a,, b)$ in G.

Case 2. $d(a) = 2(n-1)$. Then $|E(G\backslash a)| = (n-2)^2 + 1$ and hence $G\backslash a$ is Hamiltonian by Corollary 1. Let (b, z) be an edge in a Hamiltonian circuit of $G\backslash a$. Then $h(a, z,, b)$ is a Hamiltonian path in G from a to b. This proves Theorem 2.

We are now able to state

THEOREM 3. *Let $G = (X, E)$ be a strongly connected* 1-*graph of order $n > 2$ with at least $n^2 - 3n + 5$ edges. Then G is Hamiltonian.*

Proof. For $n = 3$ the theorem is clear. We assume $n > 3$. If $d(x) \geqslant n$ for all $x \in X$, then this is Theorem GH. We therefore assume the existence of a vertex x_0 such that $d(x_0) \leqslant n - 1$. Then $|E(G\backslash x_0)| \geqslant n^2 - 4n + 6$ and hence $G\backslash x_0$ is pan-Hamiltonian by Theorem 2. But G is strongly connected and hence x_0 has an outgoing as well as an incoming edge. Let them be (x_0, x_1) and (x_2, x_0) respectively. Since $G\backslash x_0$ is pan-Hamiltonian, we have a Hamiltonian circuit $(x_0, x_1,, x_2, x_0)$ in G. This proves Theorem 3.

Corollary 1 is exact for every n; Theorems 2 and 3 are exact for every $n \geqslant 3$.

APPENDIX

We now prove the theorem of Erdös and Gallai. For $c = 2$ the theorem is clear. Let the theorem be true for $c' < c$ and every n. For $n \leqslant c$ the theorem is obvious. Let the theorem hold for c and all graphs with less than n vertices. Consider a graph G of order n with $q > \frac{1}{2}(n-1)c$ edges. If G has no cycles of length $\geqslant c$, then by the induction hypothesis we have $q \leqslant \frac{1}{2}(n-1)(c-1) < \frac{1}{2}(n-1)c$, a contradiction. Then G has a cycle of length $\geqslant c$. If G has no c-cycle, there is nothing to prove. Suppose G has a c-cycle but no cycles of greater length. Let (x, y) be an arbitrary edge of G. Suppose (x, y) belongs to k triangles, $0 \leqslant k < [c/2]$, where $[t]$ denotes the greatest integer $\leqslant t$. Contract (x, y) to a vertex x_0 and consider the modified graph G_0 in which a vertex is adjacent to x_0 if and only if it was adjacent in G to x or to y. Let q_0 denote the number of edges in G_0. Since G_0 has fewer vertices than G and cycles of length at most c, we have by the induction hypothesis $q_0 \leqslant \frac{1}{2}(n-2)c$. Then

$$q = q_0 + 1 + k \leqslant q_0 + [c/2] \leqslant \tfrac{1}{2}(n-1)c - c/2 + [c/2] \leqslant \tfrac{1}{2}(n-1)c,$$

a contradiction. Then every edge belongs to at least $[c/2]$ triangles and hence every vertex is of degree at least $[c/2] + 1$.

Now consider a subgraph g spanned by a c-cycle of G. Let (x, y) be an edge of g and let (x, y, z) be a triangle in G. If $z \notin g$, then the subgraph spanned by $g \cup z$ would contain a $(c + 1)$-cycle, a contradiction. Then $z \in g$ and hence $d_g(x) \geqslant [c/2] + 1$ for every $x \in g$.

Suppose that every vertex of $G \backslash g$ is adjacent to at most one vertex of g. Contract g to a vertex x_0. The so modified graph G_0 has less than n vertices and cycles of length at most c. Let q_0 be the number of edges of G_0. Then by the induction hypothesis we have $q_0 \leqslant \frac{1}{2}[n - (c - 1) - 1]c$ so that $q \leqslant q_0 + \binom{c}{2} \leqslant \frac{1}{2}(n - 1)c$, a contradiction. Then there is a vertex v of $G \backslash g$ which is adjacent to at least two vertices of g.

Now consider the subgraph g' spanned by $g \cup v$. By a theorem of Pósa [4], g' is Hamiltonian. Since g' is of order $> c$, we arrive at a contradiction. This proves the theorem.

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